Existence and concentration of solutions of some fully nonlinear equations

Filomena Pacella

Sapienza Università di Roma

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Consider fully nonlinear elliptic equations of the type

$$\begin{cases} -F(x, D^2u) = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{FNE}$$

where

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where

- Ω smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $p > 1$ (other nonlinearities $f(x, u)$ with growth controlled by some power)
- $F = F(x, M)$, $M \in \mathcal{S}_N$ = space of $N \times N$ symmetric matrices, $x \in \Omega$, is uniformly elliptic, i.e.

$$\lambda\, Tr(P) \leq F(x, M + P) - F(x, M) \leq \Lambda\, Tr(P)$$

for some constants $0 < \lambda \leq \Lambda$ and any $x \in \Omega$, $M, P \in \mathcal{S}_N$, $P \geq 0$

The uniform ellipticity is equivalent to

$$\mathcal{M}^-_{\lambda,\Lambda}(M-P) \leq F(x,M) - F(x,P) \leq \mathcal{M}^+_{\lambda,\Lambda}(M-P)$$

for any $x \in \Omega$, $M, P \in \mathcal{S}_N$.

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$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} Tr(AM) = \lambda \sum_{\mu_i > 0} \mu_i + \Lambda \sum_{\mu_i < 0} \mu_i$$

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} Tr(AM) = \Lambda \sum_{\mu_i > 0} \mu_i + \lambda \sum_{\mu_i < 0} \mu_i$$

where $\mathcal{A}_{\lambda,\Lambda} = \{A \in \mathcal{S}_N \, : \, \lambda I_N \leq A \leq \Lambda I_N\}$, ($I_N$ identity matrix), and $\mu_1, \ldots, \mu_N$ are the eigenvalues of the matrix $M \in \mathcal{S}_N$

- Pucci's extremal operators act as barriers for the whole class of uniformly elliptic operators
- They play a crucial role in the regularity theory for fully nonlinear elliptic equations [Caffarelli-Cabré, AMS book 1995]
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- They can be seen as a generalization of the Laplace operator

$$\Delta(\cdot) = Tr(D^2 \cdot)$$

In particular we could consider the problem

$$\begin{cases} -\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{PE}$$

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Comparison with the extensively studied *Lane-Emden problem*

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{LE}$$

- ... Crucial differences but also some similarities !

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Moving plane method (Alexandrov; Serrin; Gidas-Ni-Nirenberg), which relies on maximum principles, used to get symmetry results works also for solutions of (FNE) ([Da Lio-Sirakov 2007], [Birindelli-Demengel 2013]).

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Also other kind of symmetry results in the same spirit of those obtained in [P. 2002] and [P.-Weth 2007] via *Morse index* can be proved for solutions of (FNE) ([Birindelli-Leoni-P. 2015]) because they rely on maximum principles.

Existence of solutions of (FNE)

If F is $\mathcal{M}^{\pm}_{\lambda,\Lambda}$, an existence result for positive/negative solutions in general smooth bounded domains [Quaas-Sirakov 2011] holds under the "*subcritical*" assumption

$$p \leq p^{+} = \frac{\tilde{N}_{+}}{\tilde{N}_{+} - 2}\,, \quad \tilde{N}_{+} = \frac{\lambda}{\Lambda}(N-1)+1 \quad \left(\text{for } \mathcal{M}^{+}_{\lambda,\Lambda}\right)$$

$$p \leq p^{-} = \frac{\tilde{N}_{-}}{\tilde{N}_{-} - 2}\,, \quad \tilde{N}_{-} = \frac{\Lambda}{\lambda}(N-1)+1 \quad \left(\text{for } \mathcal{M}^{-}_{\lambda,\Lambda}\right)$$

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Note that when $\lambda = \Lambda$ then $p^{+} = p^{-} = \frac{N}{N-2}$ is the so-called *Serrin* exponent.

Proof based on a fixed point theorem and relies on a-priori estimates which, in turn, derive from Cauchy-Liouville type nonexistence results in $\mathbb{R}^N$ or in the half space $\mathbb{R}^N_+$ through a blow-up procedure ([Cutrì-Leoni 2000], [Quaas-Sirakov 2011]).

It can be extended to more general uniformly elliptic fully nonlinear equation ([Armstrong-Sirakov 2011]).
A more precise result has been obtained in the *radial case* [Felmer-Quaas 2003] in the ball. They prove the existence of a critical exponent p_+^* (resp. p_-^*) such that

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- for $p < p_+^*$ (resp. p_-^*) (PE) has a positive radial solution
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- for $p \geq p_+^*$ (resp. p_-^*) (PE) does not have any positive radial solution
- p_+^* (resp. p_-^*) is not explicitly known but:

$$\frac{N+2}{N-2} < p_+^* < \frac{\tilde{N}_+ + 2}{\tilde{N}_+ - 2}$$

$$\frac{\tilde{N}_- + 2}{\tilde{N}_- - 2} < p_-^* < \frac{N+2}{N-2}$$

(p_+^* related to $\mathcal{M}_{\lambda,\Lambda}^+$ and p_-^* related to $\mathcal{M}_{\lambda,\Lambda}^-$)

p^*_+ and p^*_- can be characterized as the only exponents $p > 1$ for which the analogous of (PE) in the whole $\mathbb{R}^N$ admits a positive fast decaying radial solution U^+ (resp. U^-), i.e.

$$U^{\pm} \overset{r\to\infty}{\longrightarrow} 0 \quad \text{as} \quad \frac{1}{r^{\tilde{N}_{\pm}-2}}$$

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Remark

Note that sign changing solutions cannot be treated in the same way as the one-sign solutions.
This is observed also at eigenvalues level.

THE SEMILINEAR CASE

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{S}$$

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No solutions (neither positive/negative or sign changing) if

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Existence of solutions in any bounded domains if

$$p < \frac{N+2}{N-2} \quad (N \geq 3), \quad p > 1 \quad (N = 2)$$

or

$$p = \frac{N+2}{N-2}, \quad \Omega \textit{ nontrivial topology}$$

$$p > \frac{N+2}{N-2}, \quad \text{some domains } (\textit{holes})$$

Semilinear problem is variational so the bound on the exponent is related to the *lack of compactness for the Sobolev embedding*

$$H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)\,, \quad 2^* = \frac{2N}{N-2}$$

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which does not allow to use standard variational methods to produce solutions.

However topological or geometrical conditions on Ω can change the situation (fundamental contribution by Bahri-Coron).
In domains with nontrivial topology there exists at least a positive solution even if

$$p = \frac{N+2}{N-2} = 2^* - 1$$

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In particular if

$$\Omega = A_{a,b} = \left\{ x \in \mathbb{R}^N \,:\, 0 < a < |x| < b \right\}$$

the *compact embedding* of the space

$$H^1_{0,\mathrm{rad}}(A_{a,b}) = \{ u \in H^1_0(A_{a,b}) \,:\, u \text{ is radial} \}$$

into $L^p_{\mathrm{rad}}(A_{a,b})$ for any $p > 1$ implies the existence of a positive/negative solution for every $p > 1$.

And also the existence of ∞ many *sign changing radial solutions* can be proved $\forall p > 1$.

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Q3 Does the topology or geometry of Ω have any relation with existence or nonexistence of solutions of (FNE)?

Questions

- Q1 Are there (positive/negative) solutions beyond the "critical" exponents p^*_+ or p^*_-?
- Q2 Existence of sign changing solutions? Multiplicity? Infinitely many? (Also in the subcritical case)
- Q3 Does the topology or geometry of Ω have any relation with existence or nonexistence of solutions of (FNE)?
- Q4 Does a concentration phenomenum appear in approaching the "critical exponent"?

Theorem 1 [Galise-Leoni-P. 2016-2017]

If F is radially symmetric and $F(x,0) \equiv 0$ then in any annulus $A_{a,b}$ the fully nonlinear problem

$$\begin{cases} -F(x, D^2u) = |u|^{p-1}u & \text{in } A_{a,b} \\ u = 0 & \text{on } \partial A_{a,b} \end{cases}$$

has a positive and a negative radial solution for any $p > 1$.

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has a positive and a negative radial solution for any $p > 1$.

$\diamond$ Proof relies on careful study of the associated ODE problem (easier if p subcritical, but not obvious if $p \geq$ critical) and the maximum principle.

In our case by the uniform ellipticity condition we reduce to study the following differential inequalities :

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}\left(\frac{u'(r)}{r} I_N + \left(u''(r) - \frac{u'(r)}{r}\right) e_1 \otimes e_1\right) \leq u^p(r) \\ -\mathcal{M}^-_{\lambda,\Lambda}\left(\frac{u'(r)}{r} I_N + \left(u''(r) - \frac{u'(r)}{r}\right) e_1 \otimes e_1\right) \geq u^p(r) \end{cases}$$

(eigenvalues of the Hessian matrix are $u''(r)$ which is simple and $\frac{u'(r)}{r}$ which has multiplicity $N-1$).

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(eigenvalues of the Hessian matrix are $u''(r)$ which is simple and $\frac{u'(r)}{r}$ which has multiplicity $N-1$).

So, according to the monotonicity and the convexity of u we can distinguish three different cases:

C_1: $u'(r) \geq 0$ and $u''(r) \leq 0$, so that u satisfies

$$\begin{cases} -\lambda\, u''(r) - \Lambda(n-1)\dfrac{u'(r)}{r} \leq u^p(r) \\ -\Lambda\, u''(r) - \lambda(n-1)\dfrac{u'(r)}{r} \geq u^p(r) \end{cases}$$

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C_2: $u'(r) \leq 0$ and $u''(r) \leq 0$, so that u satisfies

$$\begin{cases} -\lambda \left(u''(r) + (n-1)\dfrac{u'(r)}{r} \right) \leq u^p(r) \\ -\Lambda \left(u''(r) + (n-1)\dfrac{u'(r)}{r} \right) \geq u^p(r) \end{cases}$$

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C₃: $u'(r) \leq 0$ and $u''(r) \geq 0$, so that u satisfies

$$\begin{cases} -\Lambda\, u''(r) - \lambda(n-1)\dfrac{u'(r)}{r} \leq u^p(r) \\ -\lambda\, u''(r) - \Lambda(n-1)\dfrac{u'(r)}{r} \geq u^p(r) \end{cases}$$

Idea of the proof

As a consequence of the uniform ellipticity the associated ODE can be written in normal form as:

$$\begin{cases} u''(r) = \mathcal{G}\left(r, \frac{u'(r)}{r}, -u^p(r)\right) & \text{if } a < r < b \\ u(r) > 0 & \text{if } a < r < b \\ u(a) = u(b) = 0 \end{cases}$$

with $\mathcal{G}(r, \cdot, \cdot)$ *uniformly Lipschitz continuous*.

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As a consequence of the uniform ellipticity the associated ODE can be written in normal form as:

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with $\mathcal{G}(r, \cdot, \cdot)$ *uniformly Lipschitz continuous*.

Then we consider initial value problem

$$\begin{cases} u''(r) = \mathcal{G}\left(r, \frac{u'(r)}{r}, -u^p(r)\right) & \text{if } r > a \\ u(a) = 0, \quad u'(a) = \alpha & \end{cases} \tag{IVP}$$

α is the shooting parameter.

(IVP) has a unique positive solution $u(r, \alpha)$ defined on a maximal interval $[a, \varrho(\alpha))$, $\varrho(\alpha) \leq +\infty$.

? Prove that $\forall p > 1$ and for any a, b with $0 < a < b$ there exists $\alpha > 0$ such that $\varrho(\alpha) = b$?

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Preliminary question:

? Are there α's for which $\varrho(\alpha) < +\infty$?

Remark

Note that if $\varrho(\alpha) = +\infty$ it means that there is a positive (super-)solution in the unbounded domain given by the exterior of the ball of radius $a > 0$, for a problem involving $\mathcal{M}^-_{\lambda,\Lambda}$.

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Note that if $\varrho(\alpha) = +\infty$ it means that there is a positive (super-)solution in the unbounded domain given by the exterior of the ball of radius $a > 0$, for a problem involving $\mathcal{M}^-_{\lambda,\Lambda}$.
But [Armstrong-Sirakov 2011] proved a nonexistence result if p subcritical. So the subcritical case is easier because at least we know that $\varrho(\alpha) < +\infty$ $\forall \alpha > 0$.

If $p > 1$ is any exponent we have to study carefully the function $\varrho(\alpha)$ and also the maximum point $\tau(\alpha)$ and the maximum value $u(\tau(\alpha), \alpha)$, as α varies.
We succeed in proving (maximum principle, principal eigenvalues) that $\forall p > 1$ there exists α^* such that for $\alpha > \alpha^*$ the radius $\varrho(\alpha)$ is finite. So that the set

$$D = \{\alpha \in (0, +\infty) \ : \ \varrho(\alpha) < +\infty\}$$

contains an unbounded connected component $(\alpha^*, +\infty)$ which can be proved to be sent onto the interval $(a, +\infty)$ by the continuous function $\varrho(\alpha)$. $\blacksquare$

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Theorem 2 [Galise-Leoni-P. 2016-2017]

In any annulus $A_{a,b}$, for any $p > 1$ and for any $k \in \mathbb{N}$ there exist two solutions u_k^+ and u_k^- of (FNE) having precisely k nodal regions ($u_k^+(0) > 0$, $u_k^-(0) < 0$).

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⋄ The proof of this result is done by induction showing that it is possible to divide the interval $[a, b]$ in k sub-intervals

$$\left[a = r_{k,0}^+, r_{k,1}^+\right], \ \left[r_{k,1}^+, r_{k,2}^+\right], \dots, \ \left[r_{k,j-1}^+, r_{k,j}^+ = b\right]_{j=1,\dots,k}$$

in such a way that choosing in each annulus $A_{r_{k,j-1}^+, r_{k,j}^+}$ the positive or negative solution found before (in an alternate way) the resulting function u_k^+ is a classical smooth solution of the fully nonlinear problem (FNE).

The same "gluing" method together with a rescaling argument can be also used in any ball B_R if p is *subcritical*.

Theorem 3 [Galise-Leoni-P. 2016-2017]

If $\Omega = B_R$, $F = F(M)$ is positively homogeneous and radially symmetric, and p subcritical, then for any $k \in \mathbb{N}$ there exist two solutions u_k^+ and u_k^- of (FNE), having exactly k nodal regions ($u_k^+(0) > 0$, $u_k^-(0) < 0$).

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If $\Omega = B_R$, $F = F(M)$ is positively homogeneous and radially symmetric, and p subcritical, then for any $k \in \mathbb{N}$ there exist two solutions u_k^+ and u_k^- of (FNE), having exactly k nodal regions $(u_k^+(0) > 0,\ u_k^-(0) < 0)$.

The exponent is *subcritical* because we construct the sign changing solutions gluing the one-sign solution in the ball which exists for p subcritical and the radial solutions in the annuli we have got in Theorem 1.

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Consider the problem

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B is the unit ball in $\mathbb{R}^N$ and $u^\pm_\varepsilon$ the unique positive solution for $p_\varepsilon = p^*_\pm - \varepsilon$.

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- $\exists!\ r_0(\varepsilon) \in (0,1)$ such that $u_\varepsilon''(r_0(\varepsilon)) = 0$
- $u_\varepsilon''(r) < 0$ for $r \in (0, r_0(\varepsilon))$ and $u_\varepsilon''(r) > 0$ in $(r_0(\varepsilon), 1)$

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- $\exists!\ r_0(\varepsilon) \in (0,1)$ such that $u''_\varepsilon(r_0(\varepsilon)) = 0$
- $u''_\varepsilon(r) < 0$ for $r \in (0, r_0(\varepsilon))$ and $u''_\varepsilon(r) > 0$ in $(r_0(\varepsilon), 1)$
- $u'_\varepsilon(r) < 0\ \forall r \in (0,1])$, $u'_\varepsilon(0) = 0$ and $u_\varepsilon(0) = \|u_\varepsilon\|_\infty = M_\varepsilon$

$(u_\varepsilon$ is u^+_ε or $u^-_\varepsilon)$

We have [Birindelli-Galise-Leoni-Pacella, work in progress]
:

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- $u_\varepsilon \to 0$ locally uniformly in $B \setminus \{0\}$ as $\varepsilon \to 0$ (for $\mathcal{M}^-_{\lambda,\Lambda}$)

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- OQ5 Uniqueness of positive/negative radial solution in an annulus (some results by [Birindelli-Galise-Leoni]).